PACKING TRIANGLES ON A SPHERE

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Abstract. The question of how many regular unit tetrahedra with a vertex at the origin can be packed into the unit sphere is a well-known and difficult problem. The answer is between 20 and 22, and these results are reproduced-here. The classical method for calculating such bounds involves projecting the tetrahedra onto the surface of the sphere, yielding equilateral spherical triangles. Motivated by this, we search for upper and lower bounds on how many equilateral spherical triangles may be arranged on the unit sphere as a function of the spherical angle α between two sides. For $\alpha \geq \frac{2\pi}{5}$, we describe some possible arrangements of equilateral spherical triangles on the sphere, but the primary focus of this paper is on $\frac{\pi}{3} < \alpha \leq \frac{2\pi}{5}$, i.e., on arbitrarily small triangles. We exhibit an algorithm that results in a packing of $\Theta(n)$ spherical triangles of area $\frac{4\pi}{n}$, within a constant fraction of the best possible. Note that $\Theta(n)$ is the conventional asymptotic notation for sphere packing problems.

1. Introduction

Our investigations begin with the following well-known problem:

Open Question 1.1. *How many regular tetrahedra of side length* 1 *may be packed into the unit sphere (the sphere of radius 1) without overlap, except on the sides, such that each tetrahedron has one vertex at the sphere's center?*

An upper bound is given by dividing the volume of the sphere by the volume of a regular tetrahedron. A better upper bound is given by projecting the tetrahedron onto the sphere's surface. To do this, we extend the sides of the tetrahedron through the sphere where they meet the sphere in geodesic segments. This turns the question from one of volume into one of surface area. A simple calculation of the edge length of these equilateral spherical triangles allows us to restate the above question as:

Open Question 1.2. *How many equilateral spherical triangles of side length $\frac{\pi}{3}$ may be packed onto the surface of the unit sphere, without overlap except at the edges?*

Definition 1.3. For $0 < \tau \leq \frac{\pi}{2}$ we define $U(\tau)$ to be the maximum number of equilateral spherical triangles of side length τ that can be arranged on the surface of the unit sphere such that each triangle intersects other triangles only at the edge.

This motivates the more general question that is our main focus:

Open Question 1.4. *How many equilateral spherical triangles of side length τ can be packed on the unit sphere, without overlap except at the edges? That is, determine the function $U(\tau)$.*

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Determining $U(\tau)$ exactly is quite difficult, so we determine upper and lower bounds instead. In particular, we derive asymptotics for $U(\tau(n))$, where $\tau(n)$ is the side length of an equilateral triangle on the unit sphere of area $\frac{4\pi}{n}$. Clearly, $U(\tau(n)) \leq n$; we will show (in Section 4) that $U(\tau(n)) = \Theta(n)$.

In Section 2, we re-derive the best upper and lower bounds known for Question 1.1. In Section 3, we use the concepts of Section 2 to define a more general procedure that we call the Windmill Algorithm for packing equilateral triangles onto a sphere. In Section 4 we prove several facts about the Windmill Algorithm and use those facts to prove a lower bound on $U(\tau)$. In Section 5 we present questions and directions for further research.

2. Upper and lower bounds for regular tetrahedra

Theorem 2.1. $20 \leq U(\frac{\pi}{3}) \leq 22$.

We split the proof into two lemmas:

Lemma 2.2. $U(\frac{\pi}{3}) \leq 22$.

Proof. To get the upper bound, we use the following standard formula (see [1])

$$\frac{\cos(\frac{1}{2}(B-C))}{\sin(\frac{1}{2}A)} = \frac{\sin(\frac{1}{2}(b+c))}{\sin(\frac{1}{2}a)} \tag{2.3}$$

where A, B, C are the angles of a spherical triangle, and a, b, c are the lengths of the sides of the triangle opposite A, B, C respectively. Since our triangles are equilateral triangles with side length τ, all three angles are equal; call their common value α. Plugging into (2.3) gives

$$\frac{1}{\sin(\frac{1}{2}\alpha)} = \frac{\sin(\tau)}{\sin(\frac{1}{2}\tau)}$$

which becomes

$$\alpha = 2\arcsin\left(\frac{1}{2\cos(\frac{\tau}{2})}\right). \tag{2.4}$$

For future purposes, it is worth writing τ in terms of α as well:

$$\tau = 2\arccos\left(\frac{1}{2\sin(\frac{\alpha}{2})}\right) \tag{2.5}$$

Substituting $\tau = \frac{\pi}{3}$ into (2.4) gives

$$\alpha = 2\arcsin\left(\frac{1}{\sqrt{3}}\right).$$

By the Gauss–Bonnet Theorem, the area of this triangle is $A_T = 3\alpha - \pi$. The surface area of the unit sphere is 4π, yielding the upper bound $\lfloor \frac{4\pi}{A_T} \rfloor = 22$. ☐

Lemma 2.6. $20 \leq U(\frac{\pi}{3})$.

Before rigorously proving this lemma, it is worth noting that these 20 equilateral spherical triangles may be packed on the unit sphere in an icosahedral arrangement. If $\alpha = \frac{2\pi}{5}$, then we can pack exactly 20 triangles on the sphere. Recalling that we originally projected tetrahedra onto the surface of the sphere; we do the reverse here and get 20 tetrahedra packed inside the sphere. These 20 regular tetrahedra form a regular icosahedron.

Note that in this lemma, $\alpha < \frac{2\pi}{5}$. We state the following lemma without proof.

Lemma 2.7. *Consider two equilateral triangles T_1 and T_2 with angles α_1 and α_2 respectively, where $\alpha_1 \leq \alpha_2$. Then T_1 can fit entirely inside T_2.*

Thus, by using this icosahedral arrangement, it is immediate that for any $\alpha < \frac{2\pi}{5}$, we can pack 20 triangles onto the unit sphere. However, the more complex proof of Lemma2.6 written below is an example of a technique that will be generalized in the following section.

Proof. To get a lower bound, note that since $\tau = \frac{\pi}{3}$, we can fit exactly 3 triangles from the top of the sphere to the bottom, corner to corner, with edges all lying along one great circle. Also note that we can fit up to five triangles around the north pole, each with one vertex at the north pole, since $5\alpha < 2\pi < 6\alpha$. We call these five triangles Row 1.

This configuration of five triangles may be duplicated on the southern hemisphere, and thus we have fit 10 triangles on the sphere so far. Next, we claim that a triangle may be placed directly adjoining each of the triangles encircling the north pole, edge to edge, so that there are now five "diamonds" connected to the north pole.

To see why this is possible, let h be the height of one of these spherical triangles (that is, the length of a perpendicular bisector). Then this half of our original triangle has sides of length $h, \tau, \frac{\tau}{2}$ and angles of magnitude $\alpha, \frac{\pi}{2}, \frac{\alpha}{2}$. We use the following formula from [1]:

$$\frac{\sin(a)}{\sin(c)} = \sin(A) \tag{2.8}$$

Plugging in $h = a, \tau = c$, and $\alpha = A$, we solve for h and get, after simplification,

$$h = \arcsin\left(\sin(\tau)\sin(\alpha)\right).$$

This may be rewritten purely in terms of α as

$$h = \arcsin\left(\cot(\alpha)\sqrt{1 - 2\cos(\alpha)}\right). \tag{2.9}$$

Plugging in $\alpha = 2\arcsin\left(\frac{1}{\sqrt{3}}\right) \approx 1.2309594$ gives $h \approx .955316574$ radians.

It is easily checked that $h \leq \tau$ for all $\alpha \in (\frac{\pi}{3}; \frac{\pi}{2}]$, as follows. Applying Equation (2.8) with $h = a, \tau = c, \alpha = A$ gives

$$\frac{\sin(h)}{\sin(\tau)} = \sin(\alpha).$$

Since $\sin(\alpha) \leq 1$, it follows that $\sin(h) \leq \sin(\tau)$. Because $\sin(x)$ is an increasing function on the interval $(\frac{\pi}{3}; \frac{\pi}{2}]$, it follows that $h \leq \tau$ for all $\alpha \in (\frac{\pi}{3}; \frac{\pi}{2}]$.

Note that because $h \leq \tau$, five triangles may be placed directly below the five triangles in the first row without hitting the bottom arrangement of five triangles. Call this new row of triangles Row 2.

The angle between any two triangles in Row 2 that share a vertex is $2\pi - 4\alpha$. Therefore another triangle fits between them sharing a vertex with the other 4 triangles, two from Row 1 and two from Row 2. Additionally, this new triangle lies within the 2τ–neighborhood of the north pole. Thus, this triangle doesn't intersect the group of five triangles around the south pole. So 10 triangles can be packed

into Row 2 without overlap. Therefore, 20 triangles may be packed on the sphere, which was our desired lower bound. □

3. The Windmill Algorithm

As mentioned in the introduction, the bounds of Section 2 are the best known bounds for Question 1.1, despite the attempts of several noteworthy mathematicians. However, one may ask what kind of lower bound this packing gives as a function of τ, i.e., what lower bound this gives for Question 1.4.

We may partition the northern hemisphere of the sphere into ring-like rows along latitude lines which have width τ. This implies that the placement of equilateral spherical triangles within one row is independent of the placement of triangles in any other row. We define R to be the number of such rows in the northern hemisphere of the sphere.

Definition 3.1. The **row count**, denoted R, for an equilateral triangle of side length τ is the quantity $R = \left\lfloor \frac{\pi}{2\tau} \right\rfloor$.

First consider the $R = 0$ case, when $\tau > \frac{\pi}{2}$, and thus $\alpha > \frac{\pi}{2}$. A calculation based on area shows that at most seven such triangles can fit on the unit sphere. If $\alpha = \frac{2\pi}{3}$, then we can fit exactly four onto the unit sphere in a tetrahedral arrangement. Further analysis of the $R = 0$ case is possible, but this is not the primary focus of this paper.

Second, consider the case $R = 1$ and $\alpha > \frac{2\pi}{5}$. Then one can fit at least 8 triangles onto the sphere, since if $R = 1$ then $\tau \leq \frac{\pi}{2}$. If $\tau = \frac{\pi}{2}$ then $\alpha = \frac{\pi}{2}$, so exactly eight triangles may be packed on the unit sphere in an octrahedral arrangement. Therefore, by Lemma 2.7, if $\tau < \frac{\pi}{2}$, then eight triangles still fit on the unit sphere.

Our main focus is as $R \to \infty$. For the rest of the paper, we assume $R \geq 2$. On the first row, we can fit five triangles around the north pole. Using equation (2.4) with $\tau = \frac{\pi}{4}$, we calculate $\alpha \approx 1.14371774$. Thus, some surface area is unused by this arrangement of triangles. We place triangles below these five triangles in the manner previously described: base to base, and then place one triangle to the left of each triangle in the second row.

Repeating this process in each row gives a lower bound of $2(5 + 10R)$ triangles (the factor of two includes the southern hemisphere). However, this will not be an optimal bound in general. Firstly, what if we could fit another row of 10 around the equator of the sphere? This is possible when

$$\frac{\pi}{2} - R\tau = \frac{\pi}{2} - \left\lfloor \frac{\pi}{2\tau} \right\rfloor \tau \geq \frac{\tau}{2}.$$

Since this is true of the southern hemisphere too, then when considering a packing on the entire sphere, we can also fit another row of 10 around the middle of the sphere. In that case, the the number of triangles in the packing is

$$2(5 + 10R) + 10 = 20(R + 1).$$

We ask, what asymptotic lower bound does this give for $U(\tau(n))$? Note that the area of an equilateral spherical triangle of side length τ is asymptotically $\Theta(\tau^2)$. Thus, $\tau(n) = \Theta(\frac{1}{\sqrt{n}})$, and thus $R = \Theta(\sqrt{n})$. Therefore, this packing gives a lower bound that is within a constant factor of $\sqrt{n}$.

However, to get a better lower bound we examine the wasted space.

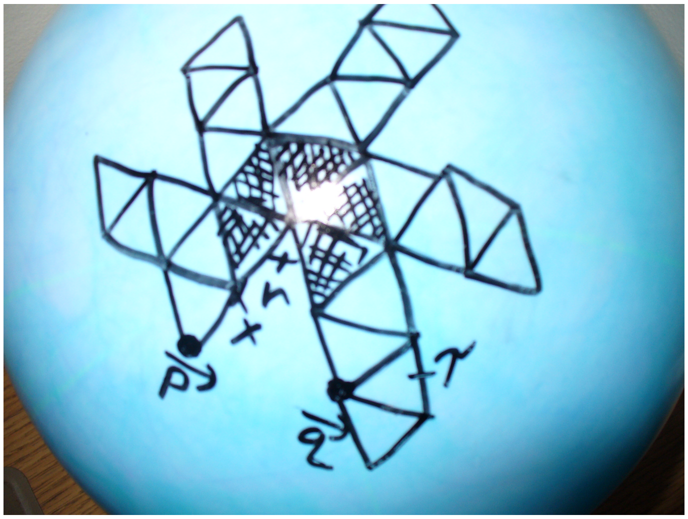

FIGURE 1. The first steps of the Windmill Algorithm

Question 3.2. *For what values of α (equivalently, what values of τ) can we fit a triangle in the area of the second row below the wasted area in the first row, and disjointly from the third row? Wasted area refers to the area of the sphere not covered by triangles.*

Answer. See Figure 1 for depictions of the variables.

Note that we can fit a third triangle into this uncovered space when $\vec{p} = \vec{q}$. By the symmetry of the situation, this forms a diamond with two opposite angles of $r = 2\pi - 5\alpha$ and two opposite angles of $x = 2\pi - 3\alpha$, and sides of length τ. We recall that A, B, C are the angles of our triangle, and a, b, c the sides, we use the formula

$$\frac{\tan\left(\frac{1}{2}(B+C)\right)}{\tan\left(\frac{1}{2}A\right)} = \frac{\cos\left(\frac{1}{2}(b-c)\right)}{\cos\left(\frac{1}{2}(b+c)\right)}, \tag{3.3}$$

and substitute $B = C = \frac{r}{2}, A = x, b = c = \tau$ to derive the equation

$$\frac{\tan\left(\frac{r}{2}\right)}{\tan\left(\frac{x}{2}\right)} = \frac{1}{\cos(\tau)}.$$

Using Chebyshev polynomials, this simplifies to

$$\frac{\cos^2(\alpha)\left(4\cos^2(\alpha) + 2\cos(\alpha) - 1\right)^2}{\left(4\cos^2(\alpha) - 2\cos(\alpha) - 1\right)^2} = \frac{(1+\cos(\alpha))^2(2\cos(\alpha)+1)^2}{(2\cos(\alpha)-1)^2}.$$

Let x_0 be the solution to the above equation. When $\alpha \leq x_0$, a triangle may be placed into the uncovered region of the second row adjacent to the uncovered region in the first row. □

It is worth noting that the above proof shows it is possible to place a triangle in the region described with one side against the left side of the triangle to its right. Of course, this triangle doesn't use all the area wasted in that region. The angles of the added triangle and the earlier triangle will not necessarily match up perfectly.

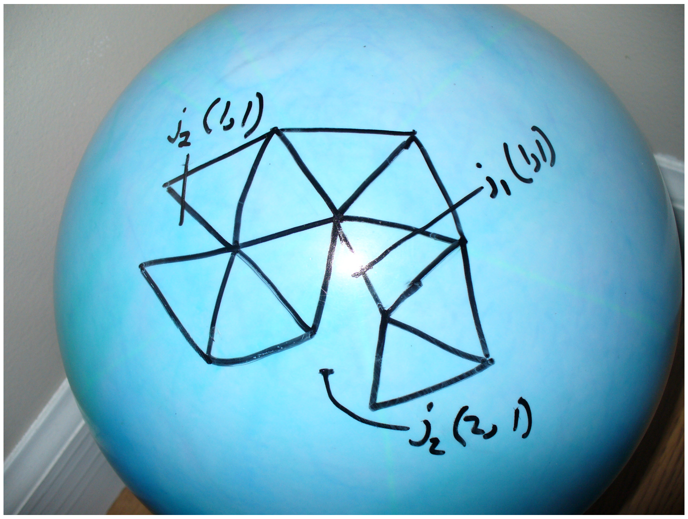

FIGURE 2. The first few types of wasted regions

How does the wasted area continue to expand in later rows? Well, there are four other congruent wasted regions in the third row congruent to the one described above, which can each have triangles placed in them in the manner we have already described. It is thus convenient to define the following notation.

Definition 3.4. We define $j_p(x, y)$ to be the area not covered by the triangles in the p–th row of the above packing, where x is the x–th development of the y–th kind of area wasted. See Figure 2 for a picture of the $j_1(1, 1), j_2(1, 1)$, and $j_2(2, 1)$ wasted areas.

Continuing this construction in the remaining rows gives rise to the following algorithm:

Algorithm 3.5 (Windmill Algorithm)**.**

(1) Group five triangles around the top of the sphere. This is the first row.
(2) Place one triangle below each triangle in the previous row, base to base.
(3) Place one triangle to the left of each triangle placed in Step 2.
(4) Go back to Step 2 until this process has been repeated R times. This creates arms projecting from the north pole; label those arms 1 through 5, where the angle at the north pole is $r = 2\pi - 5\alpha$ between arms 1 and 5 and zero between the remaining pairs of adjacent arms. (Throughout this paper, we will refer to arms as the branches of triangles placed to the left and below triangles in each row. As seen in Figure 1, such a process produces an 'arm-like' structure). Once this has been done R times, proceed to Step 5.
(5) Take the indices of the arms modulo 5. Between arms $i, i + 1$ let $u_{i,i+1}$ be the first row in which a triangle may be placed between the two arms i and $i + 1$. Place a triangle there and then place one below it.
(6) For every $k \in \mathbb{N}$ such that $u_{i,i+1} \cdot k - 1 \leq R$, place one triangle to the right followed by one to the bottom k times. We call this set of triangles a chain. Note that a chain is similar to an arm, except that a chain is formed in the area between the arms.

(7) If at any point in Step 4, Step 5 or 6 the placements would exceed the Rth row, stop the placements for that chain.

At this point, four things need mentioning. First, Step 2 and Step 3 happen in the same row. Second, this pattern makes a "windmill" with occasional triangles placed between the arms due to Steps 5 and 6. Third, there are five arms, and the behavior between arms 1 and 5 is different from the behavior between all the other arms. This is because the area between arms 1 and 5 evolves one row ahead of all the other arms, because arms 1 and 5 enclose the $j_1(1,1)$ area wasted, whereas equivalent area wasted doesn't happen between the other arms until $j_2(1,1)$ (in the second row). Fourth, these arms and the chains that branch off of them are coarse geodesics, meaning that geodesics drawn through the arms and chains (also note that these arms and chains lie within the τ–neighborhood of a geodesic). The proof of this fact is in the following section.

Example 3.6. An example of Step 5 in the Windmill Algorithm is that if $\alpha \leq x_0$, a triangle fits into the $j_2(2,1)$ area of the second row. Then a triangle can always be placed below that one (of course, this is only true for the windmill on either side of the $j_1(1,1)$ area in the first row). Thus, in the even numbered rows, we can place more sets of two triangles.

The method of the previous paragraph only works between arms 1 and 5. Between any other two arms, the method of the previous paragraph may be used, albeit delayed one row. Therefore, one could place a triangle between arms i and $i+1$ starting in the third row, and in every odd numbered row thereafter.

4. Some results regarding the Windmill Algorithm

In this section we prove several useful facts about the Windmill Algorithm.

Theorem 4.1. *Step 3 in the Windmill Algorithm is always possible, that is, a triangle can always be placed to the left of the triangles placed in Step 2.*

Proof. This proof is an argument based on symmetry. First we note that each arm may have a geodesic drawn through it. Such a geodesic is possible because the arm is symmetrical with respect to a line drawn through the middle of each triangle in an odd-numbered row. A unique geodesic G may be drawn through the centers of triangles in the first and third rows of arm 1. The arm is symmetrical, and therefore G must pass through the center of every triangle in an odd numbered row.

Consider the geodesic G_1 through arm 1 and the geodesic G_2 through arm 2. These intersect each other within a distance τ from the north pole. Because geodesics intersect exactly twice on opposite sides of the sphere, G_1 and G_2 intersect again within a distance τ of the south pole.

Thus, one can calculate that the only intersection possible between arms 1 and 2 is in the first two rows for $\alpha \leq \frac{2\pi}{5}$. No intersection occurs in the first row. By making the second row, we place one triangle below each triangle in the first row (no intersection there) and then one triangle to the left. However, because we are considering the case when five triangles can fit around a single vertex, these triangles placed to the left do not intersect any of the triangles placed below the first row. Therefore, no two arms intersect each other. □

There is one more theorem that we will use to calculate how many triangles may be fit onto the surface of the sphere using the Windmill Algorithm.

Theorem 4.2. *Suppose the first row in which triangles may be placed by step 5 of the Windmill Algorithm is Row 2. When placing a triangle in the $2n$–th row ($n \in \mathbb{N}$) during Step 6 of the Windmill Algorithm, triangles may be placed to the right and then below $n-1$ times. That is, Step 6 of the Windmill Algorithm is always possible.*

First, note that the way the theorem is phrased only holds for the space between arms 1 and 5. However, to generalize to the space between all the other arms one need only change the $2n$–th row to $(2n+1)$–th row in the statement of the theorem, and the theorem would hold. Also note that because six triangles cannot fit around a single vertex, then if a triangle T is placed to the right in Row r, and a triangle is placed below T, then no triangle may be placed to the right of that same arm in Row $r+1$.

Proof. This proof employs another symmetry argument. As we mentioned in the proof of Theorem 4.1, the arms of the windmill never touch, since the arms are coarse geodesics. No chain that branches down and to the right will ever intersect the arm it branches down from. So the only way this theorem might not hold is if the chain either hits another chain, or hits the arm opposite it. The following paragraph is a proof due to Tom Church that no two chains will ever intersect:

Consider an arm B and two chains C and D, one right below the other. These lie in the τ–neighborhoods of geodesics Y, X_1 and X_2 respectively by the same argument we made for the arms of the windmill being coarse geodesics. The geodesics X_1 and X_2 intersect Y at points p and q, respectively, inside the arm B; note that X_1 and X_2 meet Y at the same angle. Rotate the sphere by angle π around the midpoint of p and q; the symmetry of this arrangement implies that the two points where X_1 intersects X_2 are interchanged. Therefore, one such point must be on the northern hemisphere, and one on the southern hemisphere. The point on the northern hemisphere is on the other side of the arm (away from the actual chains through which the geodesic is drawn) and therefore the chains never intersect, or indeed come too close to each other, on the northern hemisphere.

We now prove that no chain starting at arm 1 (WLOG) intersects arm 5. Because the arms are coarse geodesics, and the chains of triangles to the right are coarse geodesics, we can draw a geodesic down their centers. Let X_1 be the geodesic through the chain, let Y be the geodesic through arm 1, and let Z be the geodesic through arm 5. Geodesic X_1 eventually intersects Geodesic Z, and Geodesic Y intersects Geodesic Z within τ of the north pole. However, Geodesic Y goes through $2n$ rows before intersecting Geodesic X_1, but Geodesic X_1 goes through n rows before intersecting Geodesic Z.

Let ℓ_x be the length of the segment of Geodesic X_1 that goes down n more rows. Note that the length of the segment of Y that goes down $2n$ rows is about $2nR$. Let k be the larger of the two angles created by the intersection of X_1 with Y. From [1] we have the following equation:

$$\frac{\cot(A)}{\cot(a)} = \sin(b). \tag{4.3}$$

We want to find how far to the right ℓ_x goes. Let x be the distance ℓ_x goes to the right. We thus have a right triangle of leg lengths $\approx nR, \ell_x$, and x. Substituting $A = \pi - k, a = x$, and $b = nR$ gives

$$x = \arctan(-\tan(k)\sin(nR)).$$

Let Q be the geodesic along the length of x, and let q_ℓ be the length of the segment of Geodesic Q between Y and Z. To prove the theorem, it suffices to show that $x + \frac{\tau}{\sqrt{3}} < q_\ell$. However, a simpler argument is provided here.

Let u' be the smallest angle between Y and Z formed in the τ neighborhood of the north pole. As τ gets smaller and smaller, u' approaches $\frac{\pi}{3}$ from above, and k approaches $\frac{2\pi}{3}$ from below. These two lines become approximately parallel. Therefore, for some $\tau > 0$, we would find that $x + \frac{\tau}{\sqrt{3}} \leq \frac{2}{3}q_\ell$, and thus no intersections occur.

By these arguments, no intersections occur in the Windmill Algorithm. This completes the proof of Theorem 4.2. □

It is worth noting that chains never cross the equator. The Windmill Algorithm is executed row by row, so Step 7 prevents any chain from crossing the equator.

5. A formula for a lower bound of $U(\tau)$

The Windmill Algorithm yields the following formula:

Theorem 5.1. *The total number of triangles placed on the northern hemisphere by the Windmill Algorithm is*

$$T(R) = 16 + \sum_{r=3}^{R} \left(10 + 4\left\lfloor \frac{r}{3} \right\rfloor + \left\lfloor \frac{r+1}{3} \right\rfloor \right). \tag{5.2}$$

so long as $\alpha \leq x_0$ (where x_0 is defined in Question 3.2).

Proof. In the first row we have five triangles. In the second row we have $5 \cdot 2 + 1 = 11$ triangles ($5 \cdot 2$ from Step 2 and 3 of the Windmill Algorithm, and one more from Step 5 because $\alpha \leq x_0$, so we can fit a triangle between arms 1 and 5 in the $j_2(2,1)$ section). Therefore, the total number of triangles is 16 plus the number of triangles in all the other rows.

We sum from Rows 3 to R. For each Row r, Steps 2 and 3 contribute $5 \cdot 2 = 10$ triangles. Step 6 contributes $\lfloor \frac{r}{3} \rfloor$ and that is multiplied by 4 because Step 6 applies to the four arms 1 through 4 in a manner different than between arms 1 and 5. Between arms 1 and 5 the wasted area evolves one row sooner, which is where the quantity $\lfloor \frac{r+1}{3} \rfloor$ comes from. The following argument proves that every three rows, the number of triangles added by Step 5 and Step 6 in the Windmill Algorithm increases by one. For this proof, we assume we are between arms 1 and 2. The following table should support the conclusion that Step 5 and Step 6 contributes $\lfloor \frac{r}{3} \rfloor$ triangles per row.

Row	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Chain 1			1	1												
Chain 2					1	2	1									
Chain 3							1	2	2	1						
Chain 4									1	2	2	2	1			
Chain 5											1	2	2	2	2	1
Chain 6													1	2	2	2
Chain 7															1	2
Total	0	0	1	1	1	2	2	2	3	3	3	4	4	4	5	5

The row number is listed across the top, and the table shows how many triangles are in each row from the n–th chain. The total number of triangles in a given row is shown at the bottom. Though the table only shows the results of Step 6 up to $n = 7$, the pattern is already clear. We now prove that this pattern continues.

Consider Row r, where $3|r$. A simple induction argument shows that chains end in the row after a row divisible by 3. Thus, there exists some Row k that has a chain that ends in Row $r+1$. Note that $k = \frac{2r}{3} + 1$. We suppose for simplicity that r is even, but this analysis will extend to odd r. If r is even, then no chain begins in Row r; also, no chain ends in Row r. Now we only need ask how many rows between k and r are odd (since those odd rows will have chains that extend past Row r). Let x denote the number of such rows. Then the number of triangles placed in Row r by Step 5 and Step 6 is $2 + 2x = 2(x + 1)$. Writing $r = 3p$, yields $k = 2p + 1$ for $p \in \mathbb{N}$. A simple induction argument shows that because r is even, $x = \frac{p}{2} - 1$. Therefore, there are p triangles in Row r.

Consider Row $r+1$. This row has one more triangle than the last row (since if $r+1$ is odd then a chain starts there). Because chains end in the row after a row divisible by 3, Row $r+1$ has one triangle fewer than the last row (the chain that ends in Row $r+1$ started in Row k). Therefore, there are p triangles in Row $r+1$.

Consider Row $r+2$. Because $r+2$ is not divisible by 3, no chains end in this row, so this row has one fewer triangle than Row $r+1$. However, $r+2$ is even, so a chain begins in this row, and thus this row has one more triangle in Row $r+1$. Therefore, there are p triangles in Row $r+2$.

Row $r+3$ is odd, so $x = \frac{p'-1}{2}$, where x is the number of even rows between Row $r+3$ and Row $k = 2r+3$, and $p' = p+1$. Thus, writing $r = 3p' = 3(p+1)$, the number of triangles added to Row $r+3$ in Step 5 and Step 6 is $2(x+1) = p+1$.

So Row $r+3$ has one more triangle than the previous row. A similar argument applies going from odd rows divisible by 3 to even rows divisible by 3. Thus, we find that Step 5 and Step 6 places $\left\lfloor \frac{r}{3} \right\rfloor$ triangles between arms 1 and 2 in Row r. Therefore, $T(R)$ is the total number of triangles that the Windmill Algorithm places in the northern hemisphere. □

One can see that $2T(R)$ triangles can be placed on the northern and southern hemispheres together. This lower bound could potentially be improved. For example, if

$$\frac{\pi}{2} - R\tau = \frac{\pi}{2} - \left\lfloor \frac{\pi}{2\tau} \right\rfloor \tau \geq \frac{\tau}{2}$$

then we can fit still more triangles around the sphere's equator by summing (5.2) up to R for the southern hemisphere, and up to $R+1$ for the northern hemisphere (it is not difficult to observe that the small invasion into the southern hemisphere that summing to $R+1$ entails wouldn't contradict any steps in our Windmill Algorithm).

Corollary 5.3. *Let $\tau(n)$ be the length of a spherical equilateral triangle of area $\frac{4\pi}{n}$. Then $U(\tau(n)) = \Theta(n)$ (recall that $\Theta(n)$ is asymptotic notation for packing triangles onto the unit sphere).*

Proof. As mentioned in Section 3, $R = \Theta(\sqrt{n})$ for triangles of side length $\tau(n)$. From Theorem 5.1, we find that $T(R) = \Theta(R^2) = \Theta(n)$. □

We have thus found a lower bound on the number of triangles that will cover the surface of the sphere that is linear in n, the lame upper bound.

6. Further Questions

Some further questions include:

(1) By making R a more accurate estimation of row width, we could get a better lower bound. What is a better estimate for R? Perhaps $R' = \lfloor \frac{\pi}{p'} \rfloor$, where p' is the distance from the north pole to the far corner k of a triangle placed in Step 3 of the Windmill Algorithm?

(2) At some point, the angle $r = 2\pi - 5\alpha$ will be very close to α. When would it be better to just shove a triangle into the $j_1(1,1)$ wasted area? After finding a more accurate value for R, one could say that shoving a triangle into $j_1(1,1)$ is more economical only when the triangle sticking out of the $j_1(1,1)$ area is still contained within the first row? Would this yield a better lower bound on $U(\tau)$ then the one found in this paper?

(3) The assumption that our triangles are equilateral had a valuable consequence: τ was a function of α. However, one could generalize to all sorts of triangles. What would be an analogous algorithm to the Windmill Algorithm for packing such triangles on the sphere (one could do such an analysis with isosceles triangles, for example)? How could triangles now be packed into the wasted area $j_m(x,y)$?

(4) The primary focus of this paper was how to place triangles in $j_p(x,1)$. We never explored how the wasted area could evolve beyond the case $y = 1$ in the $j_p(x,y)$ function. How would such wasted area evolve through the rows? Could a better lower bound be achieved by an analysis of this evolution?

Though many questions remain to be answered, we have found an algorithm which gives an asymptotically optimal lower bound on $U(\tau)$. Obviously, this analysis is not *a priori* helpful in answering Question 1.1. However, it is cool that our lower bound is asymptotically linear and hence it can only be improved by getting the constant closer to 1.

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